Complex Diffusion Monte-Carlo method: tests by the simulations of $2D$ electron in magnetic field and $2D$ fermions-anyons in parabolic well

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Abstract

We propose a new Complex Diffusion Monte Carlo (CDMC) method for the simulation of quantum systems with complex wave function. In CDMC the modulus and phase of wave function are simulated both in contrast to other methods. We successfully test CDMC by the simulation of the ground state for $2D$ electron in magnetic field and $2D$ fermions-anyons in parabolic well.

1 Introduction

In the quantum mechanics, there are several cases where the wave function is essentially complex(has imaginary part). The examples are $2D$ system of electrons in the external uniform magnetic field when vector potential has a central-symmetrical form and system of anyons. The simulation of these systems is impossible by using well-known Green Function Monte Carlo (GFMC) method (see review [1]) because GFMC requires the reality of wave function which is considered as the probability weight during stochastic process.

There have been several attempts [2, 3, 4] to construct a Monte Carlo method for the simulation of quantum systems with complex wave function. Authors of Ref.[2] tried to apply GFMC to this problem and met essential difficulties. Another method [4] has been developed recently by using an algorithm without branching that gave the increasing of the systematical statistical errors as a function of the simulation time.

The basic difficulty of numerical simulations of fermions is essentially the same as for the systems with complex wave function. Their wave function can change the sign and therefore, can not be used also as the probability weight. For the simulation of fermionic systems a widely used fixed node Monte

Carlo method has been developed [5] (see also [6]). Recently the constrained path Monte Carlo method was proposed[7]. In both methods one assumes the restriction on the random walks connected with the uncertainty in the space localization of the wave function node surface. CDMC method is very close to the fixed phase diffusion Monte Carlo method [3] which was applied to $2D$ electrons in magnetic field. In the framework of diffusion Monte Carlo method one simulates only modulus of the system wave function. The phase of this wave function is not simulated but treated to be fixed and equal to the phase of Laughlin's wave function [8]. Here we propose a new CDMC method *including also the simulation of the phase of the system wave function* [9, 10]. In the $2D$ space we have unique tool – so-called anyons, particles (bosons or fermions) with additional gauge interaction which provide required statistical properties [11]. So, by tuning the coupling constant we may arrive to the fermions starting from the bosons. It looks very attractive to apply the anyons for the simulation of $2D$ fermions because statistical vector potential that gives anyonic property is relatively smooth and continuous [11, 12]. The remaining main problem is that the wave function of anyonic system contains an imaginary part.

The first test of CDMC reproduces the ground state energy and its degeneracy on the orbital quantum number m of electron in magnetic field. Also the simulated wave function is in exact correspondence with analytical prediction (Fig.(1)). As the second test of CDMC, we simulate ground state energy for $2D$ fermions and anyons in parabolic well. The system of $2D$ fermions has a well-known analytical solution for this energy as a function of the number of fermions. The simulation is done for the fermionic systems with the number of particles from two to ten. We have found a good agreement of the numerical results with analytical ones for the numbers of the particles great than four (Tables (1), (2)). This observation of simulation allows us to hope that the CDMC is a good tool for the simulation of $2D$ fermion systems with big number of particles. The simulated ground state energy for anyons (Table (3)) we compare with variational calculation [13].

2 Ground state of $2D$ electron in magnetic field

Hamiltonian of the electron with $\vec{A} = [\vec{H}\vec{r}]/2$ (in the gauge $\vec{p}\vec{A} - \vec{A}\vec{p} = div\vec{A} = 0$) has a form:

$$\hat{H} = \frac{\vec{p}^{\,2}}{2M} + \frac{|e|}{Mc}\vec{A}\vec{p} + \frac{|e|^2}{Mc^2}\vec{A}^2. \tag{1}$$

Here $\vec{p} = -i\hbar\vec{\nabla}$, $\vec{\nabla} = (\partial/\partial x, \partial/\partial y)$.

Let us introduce a complex distribution function

$$f(\vec{r}, t) = \Psi_T^*(\vec{r})\Psi(\vec{r}, t). \tag{2}$$

Here $\Psi_T^*(\vec{r})$ is a complex conjugated trial wave function of electron in the magnetic field. Schrödinger equation for $\Psi(\vec{r},t)$ with imaginary time (expressed in $\hbar$ units) is:

$$-\frac{\partial\Psi(\vec{r},t)}{\partial t} = (\hat{H}-E_T)\Psi(\vec{r},t). \tag{3}$$

The equation for the distribution function $f \equiv f(\vec{r},t)$ is:

$$\begin{aligned} -\frac{\partial f}{\partial t} &= -D\Delta f + D\vec{\nabla}(f\ Re\vec{F}_Q(\vec{r})) + i[\vec{\nabla}(Df\ Im\vec{F}_Q(\vec{r})) - \\ &\quad -\frac{\hbar|e|}{Mc}\vec{A}\vec{\nabla}f] + (E_L(\vec{r})-E_T)f. \end{aligned} \tag{4}$$

Here we have

$$\vec{F}_Q(\vec{r}) = 2\Psi_T^{*-1}(\vec{r})\vec{\nabla}\Psi_T^*(\vec{r}), \tag{5}$$

$$E_L(\vec{r}) = \Psi_T^{*-1}(\vec{r})\hat{H}'\Psi_T^*(\vec{r}), \tag{6}$$

$$\hat{H}' = \frac{1}{2M}(\vec{p}-\frac{|e|}{c}\vec{A})^2 \tag{7}$$

where we putted $\vec{p} = -i\hbar\vec{\nabla}$, $D = \dfrac{\hbar^2}{2M}$, $\Delta = \vec{\nabla}^2$ and took into account

$$\vec{F}_Q(\vec{r}) = Re\vec{F}_Q(\vec{r}) + iIm\vec{F}_Q(\vec{r}) \tag{8}$$

when $\Psi_T^*(\vec{r})$ is complex.

Following [6], in the limit $\tau \to 0$ for the time step of equation (4) we assume $\vec{F}_Q(\vec{r}) \equiv \vec{F}_Q(\vec{r}\,')$ where $\vec{r}$ corresponds to time point $t+\tau$ and $\vec{r}\,'$ to t.

Let us introduce a new quantity

$$\vec{A}_Q(\vec{r},\vec{r}\,') = \frac{1}{2}Im\vec{F}_Q(\vec{r}\,') - \frac{\hbar|e|}{2DMc}\vec{A}(\vec{r}). \tag{9}$$

Then at $\tau \to 0$ the Green function of equation (4) has a form:

$$\begin{aligned} G(\vec{r},\vec{r}\,';\tau) &= G_1(\vec{r},\vec{r}\,';\tau)\exp\left[-\tau(E_L(\vec{r})-E_T)\right]\times \\ &\quad \times\exp\left[i\vec{A}_Q(\vec{r},\vec{r}\,')(\vec{r}-\vec{r}\,'-D\tau Re\vec{F}_Q(\vec{r}\,'))\right] \end{aligned} \tag{10}$$

where

$$G_1(\vec{r},\vec{r}\,';\tau) = \frac{\exp[D\tau\vec{A}_Q^2(\vec{r},\vec{r}\,')]}{4\pi D\tau}\exp\left[-\frac{(\vec{r}-\vec{r}\,'-D\tau Re\vec{F}_Q(\vec{r}\,'))^2}{4D\tau}\right]. \tag{11}$$

The Green function $G(\vec{r},\vec{r}\,';\tau)$ (10) and distribution function f (2) are the complex functions and related by usual integral equation:

$$f(\vec{r},t+\tau)=\int d\vec{r}\,' G(\vec{r},\vec{r}\,';\tau)f(\vec{r}\,',t). \tag{12}$$

From (12) the modulus and phase of f at $t+\tau$ are determined by the modulus and phase of Green function and of ones of f at t.

If $\Psi_T^*(\vec{r})$ is complex then the energy $E_L(\vec{r})$ is a complex too. Therefore, the last two exponents in (10) have a form:

$$\begin{aligned}&\exp\left[-\tau(ReE_L(\vec{r})-E_T)\right]\\&\times\exp\left[i\vec{A}_Q(\vec{r},\vec{r}\,')(\vec{r}-\vec{r}\,'-D\tau Re\vec{F}_Q(\vec{r}\,'))-i\tau ImE_L(\vec{r})\right].\end{aligned} \tag{13}$$

CDMC algorithm [9] is:

1. We prepare N_c initial configurations $\vec{r}$ in which the electron has the random and uniform distribution in a position.

2. We perform the quantum drift and diffusion of the particle: $\vec{r}_k=\vec{r}_k\,'+D\tau Re\vec{F}_Q(\vec{r}_k\,')+\chi$, for example, from k -th configuration. Here χ is Gaussian random number having zero mean value and the dispersion $2\sqrt{D\tau}$.

3. The transition into new space point is accepted with probability $P(\vec{r}\,'\to\vec{r},\tau)\equiv\min(1,W(\vec{r},\vec{r}\,'))$ where $W(\vec{r},\vec{r}\,')=\dfrac{|\Psi_T(\vec{r})|^2G_1(\vec{r}\,',\vec{r};\tau)}{|\Psi_T(\vec{r}\,')|^2G_1(\vec{r},\vec{r}\,';\tau)}$. $G_1(\vec{r},\vec{r}\,';\tau)$ is given by (11). If the transition of electron is accepted then in accordance with (13) it has a new phase; if not then the electron keeps its old phase.

4. After changing the electron position from k-configuration into new space point, one calculate $ReE_L(\vec{r}_k)$, $ImE_L(\vec{r}_k)$ and other quantities for measurements.

5. By using the first exponential factor in (13) we calculate the multiplicity M_k according to the formula $M_k=\exp[-\tau(ReE_L(\vec{r}_k)-E_T)]$. If M_k is not integer we add an uniformly distributed random number between 0 and 1 to it and take M_k equal to nearest integer.

6. If $M_k\neq 0$ then M_k copies of new k-th configuration are put in the list of the new N configurations which becomes the initial one at the next step τ in the integration of the diffusion equation. If $M_k=0$ then there is no k-th configuration in the list of new N configurations for the next time step τ. All quantities of interest such as $ReE_L(\vec{r}_k)$, $ImE_L(\vec{r}_k)$ etc. are multiplied by the factor M_k when we calculate their mean values.

7. We repeat the steps (2) - (6) until all N_c configurations are not overlooked and electrons on these configurations are not simulated on the displacement.

8. We calculate a mean energy and other mean quantities on the N configurations obtained at the step (6) at the time step τ in accordance with formula

(15).
9. We repeat the points (1) - (8) of algorithm an integer number of time steps τ of integration of diffusion equation. After that one determine the mean values of quantities $Re\overline{E}$, $Im\overline{E}$ and others on this integer number too. New value of E_T is redetermined in accordance with the formula $(E_T)_{new} = [(E_T)_{old} + Re\overline{E}]/2$ with the assumption $Im\overline{E} << Re\overline{E}$. The integer number of time steps τ represents one time block Δt.
10. Every time block Δt has the N_c initial configurations. The list of these configurations is filled by randomly chosen configurations from just ended time block. The random choice of configurations consists from two steps:
a) a random choice of configuration in the every time step τ in the time block Δt;
b) in the case N_c larger than number of the time steps τ in the time block Δt, other configurations are taken randomly from the last time step τ.
In this manner filled list of the N_c configurations will be initial one for the next time block Δt.
11. The big number repeating of time blocks Δt essentially decreases a correlation between configurations in neighbor time blocks and provides a right calculation of mean quantities.

The calculation of the mean quantities by complex distribution function $f(\vec{R}, t)$ at the time t of the running process is occurred by the formula [2]:

$$< F(t) > = \frac{\sum_{i=1}^{M} \exp[i\alpha(\vec{R}_i(t)] F(\vec{R}_i(t))}{\sum_{i=1}^{M} \exp[i\alpha(\vec{R}_i(t)]} \quad (14)$$

$$\approx \frac{\sum_{i=1}^{M} e^{i\alpha_G(\vec{R}_i(t), \vec{R}_j(t-\tau))} F(\vec{R}_i(t))}{\sum_{i=1}^{M} e^{i\alpha_G(\vec{R}_i(t), \vec{R}_j(t-\tau))}}. \quad (15)$$

Here $\vec{R}_i(t)$ is $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$, N is the number of particles and M is the number of configurations at time point t. The particle coordinates $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$ in (14), (15) are weighted with probability $|f(\vec{R}_i, t)|$ and $\alpha(\vec{R}_i(t))$ is a phase of the distribution function $f(\vec{R}_i, t)$and α_G is the phase of Green function (13).

The step from Eq. (14) to Eq. (15) is rather nontrivial [14] and can be proved with account the properties of the phase of Green function (13) α_G. Because at $\tau \to 0$ it has an asymptotic $\alpha_G \sim \tau^{1/2}$ (see [9]) and oscillates randomly around zero. In the following, the comparison of the results of numerical

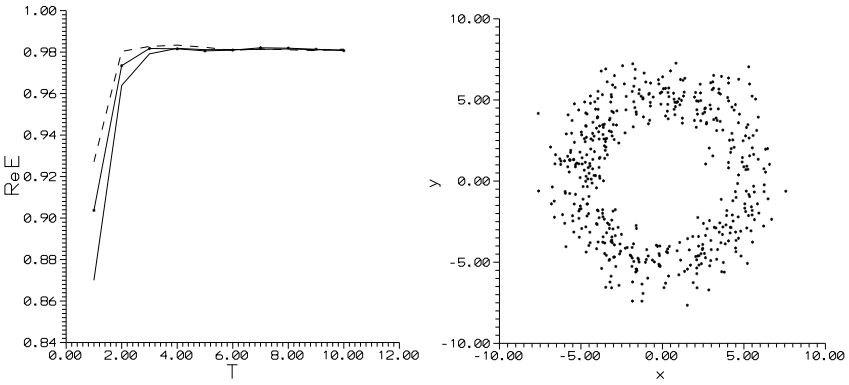

Figure 1: Left panel: The real part of the ground state energy $Re\overline{E}$ as function of the number T of time blocks Δt for different orbital quantum numbers m. Solid line, marked solid line and dashed line correspond $m = 0$, $m = 4$ and $m = 8$, respectively. Here $\alpha = \beta = \gamma = 1.0$. Right Panel: The simulation result for ground state spatial distribution of the electron in uniform magnetic field at $m = 13$. The length unity is a_c, initial number of points $N_c = 500$ and parameters $\alpha = \beta = \gamma = 1$.

calculation of ground state energy of $2D$ fermions by the formula (14) (Table (2)) and by the formula (15) (Table (1)) confirms this suggestion.

The trial wave function for the ground state of $2D$ electron in magnetic field is:

$$\Psi_T^*(\vec{r}) = \frac{C}{a_c}\left(\frac{\alpha x + i\beta y}{a_c}\right)^m \exp\left(-\gamma\frac{(x^2+y^2)}{4a_c}\right). \tag{16}$$

C - normalizing constant, $a_c = (\hbar/Mw_c)^{1/2}$ - magnetic length and $w_c = |e|H/Mc$ - cyclotron frequency, x and y - components of electron coordinate, m is the orbital quantum number, and α, β and γ are arbitrary numerical constants. When $\alpha = \beta = \gamma = 1$ (16) is an exact ground state wave function. We take a_c, $\hbar w_c/2$, and $2/\hbar w_c$ as length, energy and time units, respectively. A number of time blocks Δt is ten. For every Δt the initial number of configurations $N_c = 1000$ and number of time steps τ is equal 100. Everywhere τ was equal 0.01.

Left panel of Fig.(1) presents $Re\overline{E}$ as functions of number time blocks T. Right panel of Fig.(1) is the ground state spatial distribution of electron after simulation for $m = 13$. From this Fig. we see good correspondence with exact solution [15] for the spatial distribution which is a ring with mean radius $(2m+1)^{1/2}$ and wide 1 in a_c magnetic length units.

3 Ground state for $2D$ fermions and anyons in parabolic well

Hamiltonian of anyons in the $2D$ parabolic well has a form:

$$\hat{H} = \frac{1}{2M}\sum_{i=1}^{N}(\vec{p_i} + \vec{A}(\vec{r_i}))^2 + \frac{M\omega_o^2}{2}\sum_{i=1}^{N}\vec{r}_i^2. \tag{17}$$

Here M is the mass of particle, $\vec{p_i} = -i\hbar\vec{\nabla}_i$ where $\vec{\nabla}_i = (\partial/\partial x_i, \partial/\partial y_i)$, ω_o is a characteristic frequency of free particles in parabolic well and $\vec{r_i}$ is radius vector of i-th particle, N is number of particles.

Vector potential for anyons $\vec{A}(\vec{r_i})$ [11, 12] in (17) is

$$\vec{A}(\vec{r_i}) = \hbar\nu\sum_{j>i}^{N}\frac{\vec{z}\times\vec{r}_{ij}}{|\vec{r}_{ij}|^2}. \tag{18}$$

Here $\vec{z}$ is the unit vector perpendicular to $2D$ plane and ν - anyonic fractional statistics factor (spin of the anyon). In the bosonic representation of anyons $\nu = 0$ gives noninteracting bosons and $\nu = 1$ gives free fermions.

We introduce a complex distribution function

$$f(\vec{R}, t) = \Psi_T^*(\vec{R})\Psi(\vec{R}, t) \tag{19}$$

where $\vec{R}$ stands for the coordinates of all particles and the wave function $\Psi(\vec{R}, t)$ satisfies a Schrödinger equation (3) with hamiltonian (17).

For a bosonic representation of anyons we take (conjugated) trial wave function in the form:

$$\Psi_T^*(\vec{R}) = \prod_{i=1}^{N}\Psi_T^*(\vec{r_i}). \tag{20}$$

The distribution function $f \equiv f(\vec{R}, t)$ satisfies a diffusion equation:

$$-\frac{\partial f}{\partial t} = -D\sum_{i=1}^{N}\Delta_i f + D\sum_{i=1}^{N}\vec{\nabla}_i(f\mathrm{Re}\vec{F}_Q(\vec{r_i})) +$$
$$+i\sum_{i=1}^{N}[\vec{\nabla}_i(Df\mathrm{Im}\vec{F}_Q(\vec{r_i})) - \frac{\hbar}{M}\vec{A}(\vec{r_i})\vec{\nabla}_i f] + (E_L(\vec{R}) - E_T)f. \tag{21}$$

Here $D = \hbar^2/2M$ and $\Delta_i = \vec{\nabla}_i^2$. When a time step integration of diffusion equation (21) τ goes to zero, the Green function for this equation has a form:

$$G(\vec{R}, \vec{R}\,'; \tau) = \frac{\exp[D\tau\sum_{i=1}^{N}\vec{A}_Q^2(\vec{r_i}, \vec{r_i}\,')]}{(4\pi D\tau)^N} \times$$

$$\times \exp\left[-\frac{\sum_{i=1}^{N}(\vec{r}_i - \vec{r}_i\,' - D\tau \mathrm{Re}\vec{F}_Q(\vec{r}_i\,'))^2}{4D\tau}\right] \exp\left[-\tau(E_L(\vec{R}) - E_T)\right] \times$$

$$\times \exp\left[i\sum_{i=1}^{N}\vec{A}_Q(\vec{r}_i, \vec{r}_i\,')(\vec{r}_i - \vec{r}_i\,' - D\tau \mathrm{Re}\vec{F}_Q(\vec{r}_i\,'))\right]. \quad (22)$$

In the expressions (21) and (22) $\vec{F}_Q(\vec{r}_i)$ determines by formula (5) where radius vector and operator nabla have index i, the energy $E_L(\vec{R})$ determines by formula (6) with wave function $\Psi_T^*(\vec{R})$ and hamiltonian $\hat{H}'$ (the hamiltonian (17) with changing of sign in circle brackets (see (7) for determination of $\hat{H}'$)). $\vec{F}_Q(\vec{r}_i)$ for the complex wave function $\Psi_T^*(\vec{r}_i)$ has the real and imaginary parts.

In (22) we have introduced a new quantity

$$\vec{A}_Q(\vec{r}_i, \vec{r}_i\,') = \frac{1}{2}\mathrm{Im}\vec{F}_Q(\vec{r}_i\,') - \frac{1}{\hbar}\vec{A}(\vec{r}_i). \quad (23)$$

In the expressions (22) and (23) vectors $\vec{R}$ and $\vec{r}_i$ correspond to the time point $t + \tau$ and vectors $\vec{R}\,'$, $\vec{r}\,'$ to the time point t.

As the energy $E_L(\vec{R})$ is complex, so the expression for two last exponents in (22) has a form:

$$\exp\left[-\tau(\mathrm{Re}E_L(\vec{R}) - E_T)\right] \times$$

$$\times \exp\left[i\sum_{i=1}^{N}\vec{A}_Q(\vec{r}_i, \vec{r}_i\,')(\vec{r}_i - \vec{r}_i\,' - D\tau \mathrm{Re}\vec{F}_Q(\vec{r}_i\,')) - i\tau \mathrm{Im}E_L(\vec{R})\right]. \quad (24)$$

The ground state trial wave function for one anyon in parabolic well and in the mean magnetic field [16] with vector potential

$$\overline{\vec{A}}(\vec{r}) = \rho\pi\hbar\nu(\vec{z}\times\vec{r}) = \frac{1}{2}\vec{B}\times\vec{r} \quad (25)$$

which is generated by the average density of particles ρ can be chosen as:

$$\Psi_T^*(\vec{r}_i) = C\exp\left(-\alpha\frac{(x_i^2 + y_i^2)}{2R_o^2}\right)\exp\left(-\frac{(x_i^2 + y_i^2)}{4a_H^2}\right). \quad (26)$$

Here $\vec{B} = 2\pi\rho\hbar\nu\vec{z}$ is fictitious uniform mean magnetic field. Constant C is for normalization and $R_o = (\hbar/M\omega_o)^{1/2}$. Density ρ determines by formula $\rho = 1/\pi r_o^2$ where r_o is the mean distance between particles. We assume (see

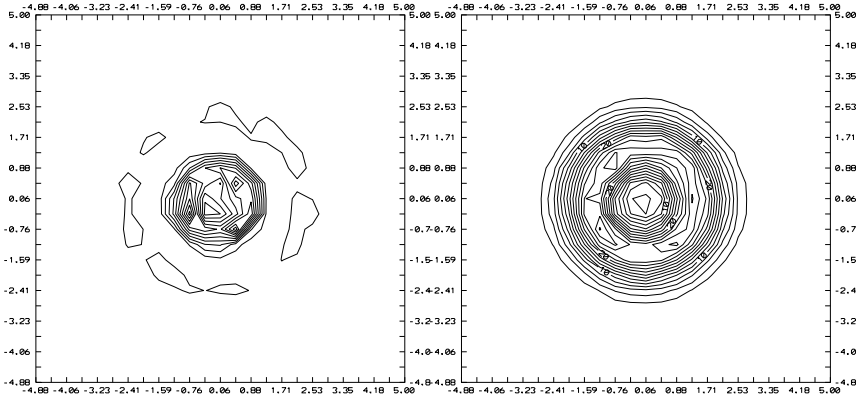

Figure 2: Square of module of one particle ground state wave function for 6 fermions $|\Psi(\vec{r})|^2 = \int |\Psi(\vec{r} = \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_6)|^2 d\vec{r}_2 d\vec{r}_3 \ldots d\vec{r}_6$. Left panel: result of the simulation. Right Panel: calculations with Slater determinant.

[10]) that r_o is equal to R_o. We express the energies in terms of $\hbar\omega_o$ and the lengths in terms of r_o. We calculate anyons ground state energy at $\nu = 0.2, 0.4, 0.6, 0.8$. The total simulation time T is equal 40 time blocks Δt. Anyons become fermions at $\nu = 1$.

For every Δt $N_c = 1000$ and the number of time steps τ is 30.

We take here optimal time step τ_0 in accordance with [10]. We find that the imaginary parts of the simulated mean quantities for all numbers fermions and anyons are essentially less than their real parts.

The results of the simulation for $2D$ fermions-anyons are given in tables (1) - (3). In tables (1) and (2) exact energies E were calculated by occupying the available lowest states of $2D$ parabolic well while the minimization of the simulated energy over parameters α gives the E_{SIM}. The comparison of the square of module of simulated one particle ground state wave function for 6 fermions in parabolic well with the square of the same module calculated by Slater determinant is given in the figures (2). From tables (1) and (2) we see that for four and bigger numbers of $2D$ fermions we have good simulation results. Good results for the simulation of ground state for the electron in magnetic field(see figures (1)) and for the systems of $2D$ fermions in parabolic well (tables (1), (2), figures (2)) (for the number of particles bigger than four) allows us to hope that CDMC provides the successful simulation of $2D$ fermionic systems with Coulomb interaction.

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N	E	E_{SIM}	$\overline{r}$	$\overline{r^2}$	α	τ_0
2	3	1.842 $\pm 1.297 \cdot 10^{-3}$	0.935 $\pm 6.994 \cdot 10^{-4}$	0.8852 $\pm 1.319 \cdot 10^{-3}$	-0.593	0.08
4	8	7.399 $\pm 4.092 \cdot 10^{-2}$	1.236 $\pm 1.844 \cdot 10^{-2}$	1.919 $\pm 4.562 \cdot 10^{-2}$	- 0.3	0.04
6	14	14.492 $\pm 3.136 \cdot 10^{-2}$	1.619 $\pm 4.261 \cdot 10^{-3}$	3.275 $\pm 1.564 \cdot 10^{-2}$	-0.092	0.02
8	22	22.383 $\pm 8.569 \cdot 10^{-2}$	2.020 $\pm 5.821 \cdot 10^{-3}$	5.112 $\pm 2.790 \cdot 10^{-2}$	-0.046	0.015
10	30	30.905 $\pm 2.010 \cdot 10^{-1}$	2.601 $\pm 2.839 \cdot 10^{-2}$	8.599 $\pm 2.095 \cdot 10^{-1}$	-0.084	0.005

Table 1: The ground state energies for the fermions in $2D$ parabolic well. Here N - numbers of particles, E - analytically calculated ground state energies, E_{SIM} - results of the simulation calculated by formula (15) (both in $\hbar\omega_o$ units), $\overline{r}$ - the simulated mean radius (in r_o units), $\overline{r^2}$ - the simulated mean square radius (in r_o^2 units), α - the numerical parameters in the wave function (26) that give a minimum E_{SIM} and τ_0 - the optimal time steps. All simulated quantities and their deviations from mean values were averaged over 30 last time blocks Δt when E_{SIM} and mean population number N_p have had relatively stable values ($N_p \approx 1000$).

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N	E	E_{SIM}	$\overline{r}$	$\overline{r^2}$	α	τ_0
2	3	1.842 $\pm 1.192 \cdot 10^{-3}$	0.935 $\pm 6.974 \cdot 10^{-4}$	0.8847 $\pm 1.418 \cdot 10^{-3}$	-0.593	0.08
4	8	7.402 $\pm 2.374 \cdot 10^{-2}$	1.242 $\pm 1.202 \cdot 10^{-2}$	1.936 $\pm 3.149 \cdot 10^{-2}$	- 0.3	0.04
6	14	14.493 $\pm 4.160 \cdot 10^{-2}$	1.612 $\pm 5.206 \cdot 10^{-3}$	3.245 $\pm 2.219 \cdot 10^{-2}$	-0.092	0.02
8	22	22.358 $\pm 6.639 \cdot 10^{-2}$	2.011 $\pm 6.379 \cdot 10^{-3}$	5.075 $\pm 3.065 \cdot 10^{-2}$	-0.046	0.015
10	30	30.896 $\pm 2.303 \cdot 10^{-1}$	2.598 $\pm 4.129 \cdot 10^{-2}$	8.574 $\pm 2.941 \cdot 10^{-1}$	-0.084	0.005

Table 2: The ground state energies for the fermions in $2D$ parabolic well. Here E_{SIM} - results of the simulation calculated by formula (14), the determinations of other quantities are the same as in table (1) and $N_p \approx 1000$.

ν	E_v	E_{SIM}	$\overline{r}$	$\overline{r^2}$	α	τ_0
0.2	32.00	30.815 ± 3.222	1.353 $\pm 7.078 \cdot 10^{-3}$	2.282 $\pm 2.430 \cdot 10^{-2}$	0.500	0.001
0.4	42.33	40.525 ± 0.268	2.081 $\pm 3.611 \cdot 10^{-2}$	5.508 $\pm 2.040 \cdot 10^{-1}$	0.500	0.001
0.6	50.59	48.371 ± 1.605	2.260 $\pm 5.124 \cdot 10^{-2}$	6.185 $\pm 2.771 \cdot 10^{-1}$	0.500	0.0075
0.8	57.68	54.119 ± 1.142	3.085 $\pm 7.307 \cdot 10^{-2}$	11.984 $\pm 6.544 \cdot 10^{-1}$	0.177	0.005
1.0	64.00	65.401 ± 0.914	3.646 $\pm 6.379 \cdot 10^{-2}$	16.872 $\pm 7.028 \cdot 10^{-1}$	-0.050	0.005

Table 3: The ground state energy as function of anyonic factor ν for 16 anyons in $2D$ parabolic well. Here ν - anyonic factor, E_v - a variational ground state energies [13] (in $\hbar\omega_o$ units) and $N_p \approx 1000$ (except $N_p \approx 2000$ at $\nu = 0.6$). The determinations of other quantities are the same as in tables (1) and (2).